On the tree-depth of random graphs

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Abstract. The tree-depth \(td(G)\) of a graph \(G\) is a measure introduced by Nešetřil and Ossona de Mendez [5] in the context of bounded expansion classes. The notion of the tree-depth is closely connected to the tree-width. The tree-width of a graph tells us how similar is \(G\) to a tree, while the tree-depth takes also into account the height of the tree.

Bounded expansion classes are defined in terms of shallow minors and its connection to tree-depth is highlighted by the following deep result. The \(k\)-th chromatic number of a graph \(\chi_k(G)\) is defined as the minimum number of colors needed to color a graph in such a way that the subgraphs \(H\) induced by any \(i\) classes of colors, \(i \leq k\), satisfy \(td(H) \leq i\). Thus \(\chi_1(G)\) is the ordinary chromatic number and \(\chi_2(G)\) is the so-called star chromatic number. The main theorem in this context states that a class of graphs \(\mathcal{C}\) has bounded expansion if and only if \(\limsup_{G \in \mathcal{C}} \chi_k(G) < \infty\) for any fixed \(k > 0\). This is a clear motivation to study tree-depth.

This parameter has been introduced under numerous names in the literature. It is equivalent to rank function [6], vertex ranking number (or ordered coloring) [2] and upper chromatic number [5].

The following inequalities relate the tree-width and tree-depth of a graph:

\[
tw(G) \leq td(G) \leq tw(G)(\log_2 n + 1)
\]  

Note that there are graphs that have bounded tree-width but unbounded tree-depth, for example trees.

To understand this new parameter, it is useful to know about its behaviour in certain classes of graphs. The main goal of this paper is to analyze how does it behave on random graphs.

We consider the ErdHos-Rényi model \(G(n, p)\) for random graph. A random graph \(G \in G(n, p)\) has \(n\) vertices and every pair of vertices is chosen independently to be an edge with probability \(p\).

If \(\mathcal{P}\) is a property that a graph can have, we will say that this property holds asymptotically almost sure (a.s.s.) for random graphs \(G \in G(n, p)\), if

\[
\lim_{n \to \infty} \Pr(G \text{ has } \mathcal{P}) = 1
\]

We will occasionally make use of the \(G(n, m)\) model of random graphs, where a graph with \(n\) vertices and \(m\) edges is chosen with the uniform distribution. As it is
well-known the two models are closely connected and a.a.s. statements are usually transferred from one model to the other one.

Our first result states the value of tree-depth for dense random graphs.

**Theorem 1.** Let $G \in G(n, p)$ be a random graph with $p = \frac{1}{o(n)}$, then $G$ satisfy a.a.s.

$$td(G) = n - o(n)$$

This theorem says that when $G$ has super-linear number of edges its tree-depth attains a really large value. Actually our proof of Theorem 1 provides the same result for tree-width. To our knowledge, the tree-width of a dense random graph has not been studied until now.

But, what happens if the number of edges is linear? This case, the sparse case, is solved by the following theorem,

**Theorem 2.** Let $G \in G(n, p)$ be a random graph with $p = \frac{c}{n}$, with $c > 0$,

1. if $c < 1$, then a.a.s. $td(G) = \Theta(\log \log n)$
2. if $c = 1$, then a.a.s. $td(G) = \Theta(\log n)$
3. if $c > 1$, then a.a.s. $td(G) = \Theta(n)$

This last theorem is closely related with a conjecture of Kloks announced in [3] on the linear behaviour of tree-width for random graphs with $c > 1$. This conjecture has been recently proved by Lee, Lee and Oum [4]. Here we give a proof of Theorem 2,(3) which, in view of inequality (1), also provides a simpler proof of Kloks conjecture. Our proof uses, as the one in [4], the same essential result of Benjamini, Kozma and Wormald [1] on the existence of an expander of linear size in a sparse random graph for $c > 1$.

**Key words:** tree-depth, tree-width, random graphs, threshold function

**References**


